V.—*On a Classification of Elastic Media, and the Laws of Plane Waves propagated through them. By the* Rev. Samuel Haughton, *Fellow and Tutor of Trinity College, Dublin.*

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In a Memoir on the equilibrium and motion of solid and fluid bodies, presented to this Academy in May, 1846, I deduced the laws of such bodies from the hypothesis of attracting and repelling molecules; since that time I have been led to consider the general laws of continuous bodies, without making any such restriction as to the nature of the molecular action. The present paper contains the results I have arrived at in this investigation, and may, perhaps, be considered interesting on account of the classification suggested as applicable to all elastic media. It consists of five sections; the first contains the general equations applicable to all media, and the properties of plane waves transmitted through them, which are readily deduced from an extension which I have given to a theorem originally stated by M. Cauchy, for a particular case; the second, third, and fourth sections contain respectively the laws of the three groups into which elastic media may be divided; these three groups consisting of,—first, bodies whose molecular action consists of exclusively normal pressures; secondly, bodies whose molecular action produces exclusively tangential forces; thirdly, bodies composed of attracting and repelling molecules. The fifth section contains a comparison of the mechanical theories of light proposed by Mr. Green and Professor Mac Cullagh, with some observations on the present state of the science of physical optics. Whatever theoretic objections may be made to the application of the theory of elastic media to optics, none such exist as to its application to solid and fluid bodies The mathematical

investigations which, in the case of light, must be hypothetical, are, in the case of solids and fluids, essentially positive, and may be made the object of direct experiment. A general inquiry into the laws of elastic media, is an interesting application of rational mechanics, and although it must necessarily include cases purely hypothetical, it is not, therefore, to be considered as unimportant. In this respect it is analogous to an inquiry into the general theory of central forces, the importance of which is not confined to the investigation of those laws, of which examples occur in nature; these are undoubtedly the most important, but the theory of central forces, considered as a branch of mechanics, would be incomplete, unless extended to all possible laws of central force.

## SECTION I.—GENERAL EQUATIONS.

The formula of virtual velocities, which contains, as shown by LAGRANGE, the conditions necessary to be fulfilled in the interior and at the boundaries of a continuous body, is the proper starting point for a deductive theory of the mechanical structure of bodies. Every hypothesis which may be made, and every fact which experience has discovered, respecting the molecular constitution of bodies, may be expressed in its most simple form by the aid of this formula; which, by its flexibility, and the facility it affords for deducing theoretic results, becomes of more importance in questions of this nature than in other mechanical problems. In order to express by means of it the conditions of equilibrium of a continuous body, it is necessary to distinguish the forces acting at each point into two classes, molecular and external forces; including among the external forces the resultants of the attractions of the other points of the body, since these attractions arise from gravitation, and must not be confounded with the molecular forces. The formula of virtual velocities must also be stated in such a manner as not to involve any hypothesis as to the nature of molecular forces, so as to possess the requisite degree of generality. If the problem be dynamical, we must then add accelerating forces equal and opposite to those actually employed, so as to destroy the motion at each point, and consider the problem as one of equilibrium of forces. These negative accelerating forces must be considered as external forces.

The forces being thus divided into two classes, the formula will consist of two parts,

$$\Sigma\,(P\delta p + P'\delta p' + \&c.) + \Sigma\,(Q\delta q + Q'\delta q' + \&c.) = 0.$$

$P$, $P'$, &c., denoting the external forces, and $Q$, $Q'$, &c., denoting the molecular forces. If $(x, y, z)$ denote the co-ordinates of a point at any instant of the motion, and a virtual displacement $(\delta x, \delta y, \delta z)$ be conceived, these quantities will be functions of four independent variables, which will be the initial values of $(x, y, z)$, and the time. Let $(u', v', w')$ denote the accelerating forces; hence

$$\Sigma\,(P\delta p + P'\delta p' + \&c.) = \iiint\{(X-u')\,\delta x + (Y-v')\,\delta y + (Z-w')\,\delta z\}\,dm.$$

I shall assume that the virtual moments of the molecular forces depend upon the differential coefficients of $(\delta x, \delta y, \delta z)$, by means of the following linear equation,*

$$\begin{aligned} Q\delta q + Q'\delta q' + \&c. = {} & P_1\frac{d\delta x}{dx} + P_2\frac{d\delta x}{dy} + P_3\frac{d\delta x}{dz} \\ & + Q_1\frac{d\delta y}{dx} + Q_2\frac{d\delta y}{dy} + Q_3\frac{d\delta y}{dz} \\ & + R_1\frac{d\delta z}{dx} + R_2\frac{d\delta z}{dy} + R_3\frac{d\delta z}{dz}. \end{aligned}$$

Inserting these values into the equation of virtual velocities, and integrating by parts, we find

$$\begin{aligned} 0 = {} & \iiint\{\rho\,(X-u')\,\delta x + \rho\,(Y-v')\,\delta y + \rho\,(Z-w')\,\delta z\}\,dxdydz \\ & \iiint\left\{\left(\frac{dP_1}{dx}+\frac{dP_2}{dy}+\frac{dP_3}{dz}\right)\delta x + \left(\frac{dQ_1}{dx}+\frac{dQ_2}{dy}+\frac{dQ_3}{dz}\right)\delta y + \left(\frac{dR_1}{dx}+\frac{dR_2}{dy}+\frac{dR_3}{dz}\right)\delta z\right\}dxdyd \\ & + \iint(P_1\delta x + Q_1\delta y + R_1\delta z)\,dydz, \qquad (1) \\ & + \iint(P_2\delta x + Q_2\delta y + R_2\delta z)\,dxdz, \\ & + \iint(P_3\delta x + Q_3\delta y + R_3\delta z)\,dxdy. \end{aligned}$$

The equations of motion being determined by the triple integrals, and the conditions at the limits by the double integrals. The equations of motion formed from this equation will be

* The ground of this assumption is the fact, that molecular forces depend upon the *relative* displacements of the particles, and not on their absolute displacements.

$$\rho(X-u') = \frac{dP_1}{dx} + \frac{dP_2}{dy} + \frac{dP_3}{dz},$$
$$\rho(Y-v') = \frac{dQ_1}{dx} + \frac{dQ_2}{dy} + \frac{dQ_3}{dz}, \qquad (2)$$
$$\rho(Z-w') = \frac{dR_1}{dx} + \frac{dR_2}{dy} + \frac{dR_3}{dz}.$$

These equations are the same as those deduced by a totally different method (which will be given presently), and involve no restriction as to the extent of the deviation of $(x, y, z)$ from their original values $(a, b, c)$.

If we suppose $x = a + \xi$, $y = b + \eta$, $z = c + \zeta$, the expression for the moments of the molecular forces will become

$$\begin{aligned} Q\delta q + Q'\delta q' + \&c. = {} & P_1\delta a_1 + P_2\delta a_2 + P_3\delta a_3 \\ & + Q_1\delta\beta_1 + Q_2\delta\beta_2 + Q_3\delta\beta_3 \\ & + R_1\delta\gamma_1 + R_2\delta\gamma_2 + R_3\delta\gamma_3; \end{aligned}$$

where

$$a_1 = \frac{d\xi}{dx}, \quad a_2 = \frac{d\xi}{dy}, \quad a_3 = \frac{d\xi}{dz};$$
$$\beta_1 = \frac{d\eta}{dx}, \quad \beta_2 = \frac{d\eta}{dy}, \quad \beta_3 = \frac{d\eta}{dz};$$
$$\gamma_1 = \frac{d\zeta}{dx}, \quad \gamma_2 = \frac{d\zeta}{dy}, \quad \gamma_3 = \frac{d\zeta}{dz}.$$

If we restrict the molecular forces by the condition

$$Q\delta q + Q'\delta q' + \&c. = \delta V,$$

we shall have the relations

$$P_1 = \frac{dV}{da_1}, \quad P_2 = \frac{dV}{da_2}, \quad P_3 = \frac{dV}{da_3},$$
$$Q_1 = \frac{dV}{d\beta_1}, \quad Q_2 = \frac{dV}{d\beta_2}, \quad Q_3 = \frac{dV}{d\beta_3},$$
$$R_1 = \frac{dV}{d\gamma_1}, \quad R_2 = \frac{dV}{d\gamma_2}, \quad R_3 = \frac{dV}{d\gamma_3},$$
$$V = F(a_1, a_2, a_3, \quad \beta_1, \beta_2, \beta_3, \quad \gamma_1, \gamma_2, \gamma_3).$$

The kind of motion which it is the object of this paper to investigate is of the kind commonly called small oscillations; and for this kind of motion it is not necessary to use the most general equations, or to consider the unknown quantities of the problem as functions of $(x, y, z, t)$. We may use, instead of $(x, y, z)$, the co-ordinates $(a, b, c)$ of the position of rest of the molecules. In fact, any differential coefficient of a function $\phi$, taken with respect to $(a, b, c)$, will be expressed by the equation,

$$\frac{d\phi}{da}=\frac{d\phi}{dx}\frac{dx}{da}+\frac{d\phi}{dy}\frac{dy}{da}+\frac{d\phi}{dz}\frac{dz}{da};$$

but, since $x = a + \xi$, $y = b + \eta$, $z = c + \zeta$, we obtain

$$\frac{dx}{da}=1+\frac{d\xi}{da}, \quad \frac{dy}{da}=\frac{d\eta}{da}, \quad \frac{dz}{da}=\frac{d\zeta}{da}.$$

Hence, neglecting quantities of the second order, we find

$$\frac{d\phi}{da}=\frac{d\phi}{dx},$$

and similarly for the other differential coefficients.

In the remaining part of this memoir (unless the contrary be expressed), I shall, therefore, consider $(x, y, z)$ as the co-ordinates of the position of equilibrium of the molecules, and $(\xi, \eta, \zeta)$ as the small displacements of the molecule; the element of the mass will be expressed by the equation $dm = \epsilon dxdydz$, where $\epsilon$ denotes the density, not considered as a function of the time, since $(dxdydz)$ denotes the original element of the volume.

Two kinds of waves can pass through such a body as water; one, a surface wave, depending on the action of gravity for its propagation; the other, such a wave as propagates sound, and does not directly depend on external forces. This latter is the kind of wave described in this paper. The equations peculiar to it will be found by omitting $(X, Y, Z)$ from the general equations; but though the external forces are not explicit in the formulæ, yet as they affect the density and structure of the body differently at different points, though they do not directly affect the wave, we must introduce them implicitly by rendering $\epsilon$ and the coefficients of $V$ functions of $(X, Y, Z)$, and therefore of $(x, y, z)$

The equation of virtual velocities thus modified will become, considering $(x, y, z)$ as the positions of rest of the molecules,

$$\iiint \epsilon \left( \frac{d^2\xi}{dt^2}\delta\xi + \frac{d^2\eta}{dt^2}\delta\eta + \frac{d^2\zeta}{dt^2}\delta\zeta \right) dxdydz = \iiint \delta V dxdydz. \tag{3}$$

As $V$ is a function of the quantities $(\alpha_1, \alpha_2, \alpha_3, \&c.)$ of a given form, we shall have

$$\begin{aligned} \delta V = & \frac{dV}{d\alpha_1}\delta\alpha_1 + \frac{dV}{d\alpha_2}\delta\alpha_2 + \frac{dV}{d\alpha_3}\delta\alpha_3 \\ & + \frac{dV}{d\beta_1}\delta\beta_1 + \frac{dV}{d\beta_2}\delta\beta_2 + \frac{dV}{d\beta_3}\delta\beta_3 \\ & + \frac{dV}{d\gamma_1}\delta\gamma_1 + \frac{dV}{d\gamma_2}\delta\gamma_2 + \frac{dV}{d\gamma_3}\delta\gamma_3 . \end{aligned}$$

Substituting this value in equation (3), and integrating by parts, we obtain

$$\begin{aligned} \iiint \epsilon \left( \frac{d^2\xi}{dt^2}\delta\xi + \frac{d^2\eta}{dt^2}\delta\eta + \frac{d^2\zeta}{dt^2}\delta\zeta \right) dxdydz = & \iiint \delta V dxdydz \\ = & \iint \left( \frac{dV}{d\alpha_1}\delta\xi + \frac{dV}{d\beta_1}\delta\eta + \frac{dV}{d\gamma_1}\delta\zeta \right) dydz \\ & + \iint \left( \frac{dV}{d\alpha_2}\delta\xi + \frac{dV}{d\beta_2}\delta\eta + \frac{dV}{d\gamma_2}\delta\zeta \right) dxdz \\ & + \iint \left( \frac{dV}{d\alpha_3}\delta\xi + \frac{dV}{d\beta_3}\delta\eta + \frac{dV}{d\gamma_3}\delta\zeta \right) dxdy \\ & - \iiint \left( \frac{d}{dx}\cdot\frac{dV}{d\alpha_1} + \frac{d}{dy}\cdot\frac{dV}{d\alpha_2} + \frac{d}{dz}\cdot\frac{dV}{d\alpha_3} \right) \delta\xi dxdydz \\ & - \iiint \left( \frac{d}{dx}\cdot\frac{dV}{d\beta_1} + \frac{d}{dy}\cdot\frac{dV}{d\beta_2} + \frac{d}{dz}\cdot\frac{dV}{d\beta_3} \right) \delta\eta dxdydz \\ & - \iiint \left( \frac{d}{dx}\cdot\frac{dV}{d\gamma_1} + \frac{d}{dy}\cdot\frac{dV}{d\gamma_2} + \frac{d}{dz}\cdot\frac{dV}{d\gamma_3} \right) \delta\zeta dxdydz. \end{aligned} \tag{4}$$

The double integrals, as usual, denote the conditions at the bounding surface, and the triple integrals give the general equations of motion in the interior. The laws of propagation depend upon the triple integrals, and the laws of reflexion and refraction depend upon the double integrals.

The equations of motion derived from (4) will be,

$$
\begin{aligned}
-\epsilon \frac{d^2\xi}{dt^2} &= \frac{d}{dx}\cdot\frac{dV}{d\alpha_1}+\frac{d}{dy}\cdot\frac{dV}{d\alpha_2}+\frac{d}{dz}\cdot\frac{dV}{d\alpha_3};\\
-\epsilon \frac{d^2\eta}{dt^2} &= \frac{d}{dx}\cdot\frac{dV}{d\beta_1}+\frac{d}{dy}\cdot\frac{dV}{d\beta_2}+\frac{d}{dz}\cdot\frac{dV}{d\beta_3};\\
-\epsilon \frac{d^2\zeta}{dt^2} &= \frac{d}{dx}\cdot\frac{dV}{d\gamma_1}+\frac{d}{dy}\cdot\frac{dV}{d\gamma_2}+\frac{d}{dz}\cdot\frac{dV}{d\gamma_3}.
\end{aligned}
\qquad (5)
$$

These equations contain the laws of propagation of every variety of wave not depending on external forces; and as the function has not been restricted by any hypothesis as to the law of molecular action, they will be the dynamical equations of propagation of sound in air, water, solids, and of light, if we adopt the undulatory hypothesis. In all these cases, the difficulty is to ascertain the correct form of $V$ peculiar to the particular case; the form being found, the coefficients must be determined by experiment for each body: so far as theory is concerned, the mechanical classification of bodies would be complete, if the form of $V$ were known for all.

The conditions to be satisfied at the limits will be different, according as the surface is fixed, free, acted on by special forces, or in contact with other bodies. As there is no difficulty in forming the equations for any of these cases, I shall here give only the conditions at the limiting surface in contact with other bodies.

Let $F(x, y, z) = 0$ be the equation of the surface passing through the positions of rest of the molecules which at any instant form the actual limiting surface $f(x, y, z, t) = 0$, hence

$$\frac{dF}{dx}dx+\frac{dF}{dy}dy+\frac{dF}{dz}dz=0;$$

and if $\lambda$, $\mu$, $\nu$, be the angles which determine the position of the normal, we shall have

$$
\begin{aligned}
\cos\lambda &= \kappa\frac{dF}{dx},\\
\cos\mu &= \kappa\frac{dF}{dy},\\
\cos\nu &= \kappa\frac{dF}{dz}.
\end{aligned}
$$

If $\omega$ be an element of the surface

$$dydz = \omega \cos \lambda = \kappa\omega \frac{dF}{dx},$$

$$dxdz = \omega \cos \mu = \kappa\omega \frac{dF}{dy},$$

$$dxdy = \omega \cos \nu = \kappa\omega \frac{dF}{dz};$$

these equations will reduce the double integrals (4) to the form

$$\begin{aligned}\Delta = &\iint\left(\frac{dV}{d\alpha_1}\cdot\frac{dF}{dx}+\frac{dV}{d\alpha_2}\cdot\frac{dF}{dy}+\frac{dV}{d\alpha_3}\cdot\frac{dF}{dz}\right)\kappa\omega\delta\xi \\ +&\iint\left(\frac{dV}{d\beta_1}\cdot\frac{dF}{dx}+\frac{dV}{d\beta_2}\cdot\frac{dF}{dy}+\frac{dV}{d\beta_3}\cdot\frac{dF}{dz}\right)\kappa\omega\delta\eta \\ +&\iint\left(\frac{dV}{d\gamma_1}\cdot\frac{dF}{dx}+\frac{dV}{d\gamma_2}\cdot\frac{dF}{dy}+\frac{dV}{d\gamma_3}\cdot\frac{dF}{dz}\right)\kappa\omega\delta\zeta.\end{aligned}$$

These will be the double integrals resulting from one body, and from them must be subtracted similar terms derived from the body which bounds the one under consideration. $F(x, y, z) = 0$ is common to both bodies when at rest, also $\xi, \eta, \zeta$, will be the same for all the bounding surface, $f(x, y, z, t) = 0$, during the motion; also $\delta\xi, \delta\eta, \delta\zeta$, are independent; hence the condition at the limits will be

$$\Delta' - \Delta'' = 0,$$

which is equivalent to three equations,

$$\begin{aligned}&\left(\frac{dV_0'}{d\alpha_1}-\frac{dV_0''}{d\alpha_1}\right)\frac{dF}{dx}+\left(\frac{dV_0'}{d\alpha_2}-\frac{dV_0''}{d\alpha_2}\right)\frac{dF}{dy}+\left(\frac{dV_0'}{d\alpha_3}-\frac{dV_0''}{d\alpha_3}\right)\frac{dF}{dz}=0, \\ &\left(\frac{dV_0'}{d\beta_1}-\frac{dV_0''}{d\beta_1}\right)\frac{dF}{dx}+\left(\frac{dV_0'}{d\beta_2}-\frac{dV_0''}{d\beta_2}\right)\frac{dF}{dy}+\left(\frac{dV_0'}{d\beta_3}-\frac{dV_0''}{d\beta_3}\right)\frac{dF}{dz}=0, \qquad (6) \\ &\left(\frac{dV_0'}{d\gamma_1}-\frac{dV_0''}{d\gamma_1}\right)\frac{dF}{dx}+\left(\frac{dV_0'}{d\gamma_2}-\frac{dV_0''}{d\gamma_2}\right)\frac{dF}{dy}+\left(\frac{dV_0'}{d\gamma_3}-\frac{dV_0''}{d\gamma_3}\right)\frac{dF}{dz}=0,\end{aligned}$$

$V'$, $V''$ denoting the functions proper to the first and second body, and $V_0$ denoting that the values of $(x, y, z)$, deduced from $F(x, y, z) = 0$, have been substituted in $V$.

To the three equations (6) must be added the self-evident geometrical

equations, which denote that the vibrating molecules at the bounding surface may be considered as belonging to either body; they are three in number,

$$\xi_0' = \xi_0'', \quad \eta_0' = \eta_0'', \quad \zeta_0' = \zeta_0''. \qquad (7)$$

Equations (6) and (7) contain the laws of reflexion and refraction of vibrations for all bodies, and are completely determinate when the form of $V$ is given for each of the bodies in contact.

Equations (5), (6), (7), are *necessary and sufficient* to determine the propagation and reflexion of waves, so far as they are connected with each other; and no mechanical theory of vibrations is correct which does not exhibit this connexion, or which assumes such laws of reflexion and refraction as contradict the laws of propagation; the connexion between these laws is no proof of the truth of any theory, but the want of a connexion would be a proof of the inconsistency of a theory.

The reduction of the general equations by the omission of the external forces may require some explanation. There are two kinds of waves, as I have stated in making the reduction, one only of which is the subject of our present inquiry. Fluid bodies, such as the atmosphere and the ocean, can propagate tidal waves depending on external forces; they are also capable of propagating waves of sound which depend directly on the molecular forces; solid bodies can only propagate the second species of waves. In considering this kind of vibration, we neglect the external forces, as they are of so much less intensity than the molecular forces, that they produce no effect on the motion; but if we suppose the motion to cease, and inquire into the state of equilibrium of the body, we should then use the general formula, which includes external forces; and, even in the case of motion, all the coefficients must be considered as variable, in consequence of the position of constrained equilibrium to which the body would return, if the motion ceased.

It is evident from an inspection of equations (5) and (6) that the differential coefficients of the function $V$, with respect to ($a_1$, $a_2$, &c.), occupy an important position in the theory of elastic media. Hitherto, I have only given to them a mathematical definition, I now proceed to explain their physical meaning, by the consideration of an elementary parallelepiped of the body. If we conceive a plane drawn in any direction in the interior of the body, and consider the parts

of the body situated at opposite sides as acting upon any element of the plane; in general, the effect of the particles at one side will be to produce a normal force, and tangential forces acting in the element; these tangential forces may be resolved into two directions at right angles to each other. Let us now conceive an elementary parallelepiped, with one corner situated at the point $(x,y,z)$ (these co-ordinates here denote the actual position of the molecule at any instant); let the forces acting on the side $(dydz)$ be denoted by $(P_1, Q_1, R_1)$, $P_1$ being normal and parallel to the axis of $(x)$, $(Q_1, R_1)$ tangential and parallel to the axes of $Y$ and $Z$; let the forces acting on the side $(dxdz)$ be $(P_2, Q_2, R_2)$, $Q_2$ being the normal force; and let the forces at the side $(dxdy)$ be $(P_3, Q_3, R_3)$, $R_3$ being the normal force. The forces acting on the opposite sides of the parallelepiped will be :—on the side opposite to $(dydz)$,

$$\left(P_1+\frac{dP_1}{dx}dx\right)dydz, \quad \left(Q_1+\frac{dQ_1}{dx}dx\right)dydz, \quad \left(R_1+\frac{dR_1}{dx}dx\right)dydz;$$

on the side opposite to $(dxdz)$,

$$\left(P_2+\frac{dP_2}{dy}dy\right)dxdz, \quad \left(Q_2+\frac{dQ_2}{dy}dy\right)dxdz, \quad \left(R_2+\frac{dR_2}{dy}dy\right)dxdz;$$

on the side opposite to $(dxdy)$,

$$\left(P_3+\frac{dP_3}{dz}dz\right)dxdy, \quad \left(Q_3+\frac{dQ_3}{dz}dz\right)dxdy, \quad \left(R_3+\frac{dR_3}{dz}dz\right)dxdy.$$

These forces, acting on the six sides of the parallelepiped, must equilibrate the forces $(Xdm, Ydm, Zdm)$, applied at the centre of the parallelepiped, and arising from external causes; hence, the equations of equilibrium will be

$$Xdm + \left(\frac{dP_1}{dx}+\frac{dP_2}{dy}+\frac{dP_3}{dz}\right)dxdydz = 0;$$

$$Ydm + \left(\frac{dQ_1}{dx}+\frac{dQ_2}{dy}+\frac{dQ_3}{dz}\right)dxdydz = 0;$$

$$Zdm + \left(\frac{dR_1}{dx}+\frac{dR_2}{dy}+\frac{dR_3}{dz}\right)dxdydz = 0;$$

or, since $\rho dxdydz = dm$, and the molecular forces must act in the direction opposite to the applied forces, including negative accelerating forces,

$$
\begin{aligned}
\rho(X - u') &= \frac{dP_1}{dx} + \frac{dP_2}{dy} + \frac{dP_3}{dz}; \\
\rho(Y - v') &= \frac{dQ_1}{dx} + \frac{dQ_2}{dy} + \frac{dQ_3}{dz}; \\
\rho(Z - w') &= \frac{dR_1}{dx} + \frac{dR_2}{dy} + \frac{dR_3}{dz}.
\end{aligned} \tag{8}
$$

These are identical with equations (2), and are true for all kinds of molecular action. In the particular case of a rigid parallelepiped, we should introduce another set of conditions arising from the equilibrium of couples. Let $(Ldm, Mdm, Ndm)$ be external couples applied to the parallelepiped; these must equilibrate the couples arising from the molecular action of the surrounding parts of the body; it is easy to see that, neglecting the small forces arising from the differential coefficients of $P_1$, $P_2$, &c., the couples round the axes of $x$, $y$, $z$, will be

$$(R_2 + Q_3)\, dydz, \quad (P_3 + R_1)\, dxdz, \quad (Q_1 + P_2)\, dxdy.$$

Hence the required conditions will be

$$
\begin{aligned}
\epsilon L &= R_2 + Q_3\,; \\
\epsilon M &= P_3 + R_1\,; \\
\epsilon N &= Q_1 + P_2\,;
\end{aligned}
$$

and, if no external couples be applied, the conditions will be,

$$
\begin{aligned}
R_2 &= Q_3\,; \\
P_3 &= R_1\,; \\
Q_1 &= P_2\,;
\end{aligned} \tag{9}
$$

since the couples $(R_2, Q_3)$, &c., must act in opposite directions.

These conditions (9) were given by M. Cauchy,* and afterwards adopted by M. Poisson.† These writers seem to have considered them as necessary for all systems; but this is not true, as equations (8) exhibit all the relations which exist between the forces and the motions produced. Equations (9) are necessary

* Exercices de Mathematiques, tom. ii. p. 47.

† Journal de l'Ecole Polytechnique, cahier xx. p. 84.

for the equilibrium of a rigid parallelepiped, and will be shown in this memoir to be satisfied by the equations of equilibrium of bodies whose molecules attract and repel each other in the direction of the line joining them; but if no supposition be made as to the nature of the molecular action, there will be no condition resulting from the equilibrium of couples; for we have no right to assume, in the equilibrium of a parallelepiped, whose elements may alter their relative position, and thus develope new forces, that the same equations hold as in the equilibrium of an isolated rigid parallelepiped, for which case six equations of condition are necessary, arising from the equilibrium of forces and of couples.

Restoring the usual signification of $(x, y, z)$ in equations (8), and omitting the external forces, we obtain

$$
\begin{aligned}
-\epsilon\frac{d^2\xi}{dt^2} &= \frac{dP_1}{dx}+\frac{dP_2}{dy}+\frac{dP_3}{dz};\\
-\epsilon\frac{d^2\eta}{dt^2} &= \frac{dQ_1}{dx}+\frac{dQ_2}{dy}+\frac{dQ_3}{dz};\\
-\epsilon\frac{d^2\zeta}{dt^2} &= \frac{dR_1}{dx}+\frac{dR_2}{dy}+\frac{dR_3}{dz}.
\end{aligned}
\tag{10}
$$

These equations correspond with (5), and by comparing them we may deduce the following relations:

$$
\begin{aligned}
&\frac{dV}{d\alpha_1}=P_1,\quad \frac{dV}{d\alpha_2}=P_2,\quad \frac{dV}{d\alpha_3}=P_3;\\
&\frac{dV}{d\beta_1}=Q_1,\quad \frac{dV}{d\beta_2}=Q_2,\quad \frac{dV}{d\beta_3}=Q_3;\\
&\frac{dV}{d\gamma_1}=R_1,\quad \frac{dV}{d\gamma_2}=R_2,\quad \frac{dV}{d\gamma_3}=R_3;
\end{aligned}
$$

from which we obtain the following theorem:

"If through any point $(x, y, z)$, three elements of planes be drawn parallel to the co-ordinate planes, the total action of the part of the body lying at one side of these planes will consist of three normal and six tangential forces; and these forces may be expressed by the differential coefficients of the function $V$, with respect to $(\alpha_1, \alpha_2, \alpha_3, \&c.)$." This theorem is of importance in classifying

bodies, as it enables us to pass directly from the resultant forces of the molecules to the form of the function which determines the laws of propagation and reflexion. Introducing these forces into the conditions at the limits (6), we obtain

$$P'_{01}\frac{dF}{dx} + P'_{02}\frac{dF}{dy} + P'_{03}\frac{dF}{dz} = P''_{01}\frac{dF}{dx} + P''_{02}\frac{dF}{dy} + P''_{03}\frac{dF}{dz};$$
$$Q'_{01}\frac{dF}{dx} + Q'_{02}\frac{dF}{dy} + Q'_{03}\frac{dF}{dz} = Q''_{01}\frac{dF}{dx} + Q''_{02}\frac{dF}{dy} + Q''_{03}\frac{dF}{dz};$$
$$R'_{01}\frac{dF}{dx} + R'_{02}\frac{dF}{dy} + R'_{03}\frac{dF}{dz} = R''_{01}\frac{dF}{dx} + R''_{02}\frac{dF}{dy} + R''_{03}\frac{dF}{dz}.$$

If the axis of $z$ be made to coincide with the normal, we shall have

$$\frac{dF}{dx} = 0, \quad \frac{dF}{dy} = 0, \quad \frac{dF}{dz} = 1.$$

Hence,

$$\begin{aligned} P'_{03} &= P''_{03}, & \xi'_0 &= \xi''_0, \\ Q'_{03} &= Q''_{03}, & \eta'_0 &= \eta''_0, \\ R'_{03} &= R''_{03}, & \zeta'_0 &= \zeta''_0. \end{aligned} \tag{11}$$

These equations at the limits are the mathematical statement of two facts, of which one is mechanical and the other geometrical.

1. That the forces, normal and tangential (arising from molecular action), acting upon an element of the bounding surface, must be equal and opposite for the two bodies in contact.

2. That the motion of the particles in the bounding surface may be considered as common to both bodies.

I shall now return to the general equation (4); let the function $V$ be divided into homogeneous parts, so that

$$V = \phi_0 + \phi_1 + \phi_2 + \phi_3 + \&c.,$$

where $\phi_0$ is constant, $\phi_1$ of the first degree with respect to $(\alpha_1, \alpha_2)$, and so of the others; then, neglecting $\phi_0$ and terms higher than the second, we obtain

$$V = \phi_1 + \phi_2,$$

$$\begin{aligned}\phi_1 = {} & A_1\alpha_1 + A_2\alpha_2 + A_3\alpha_3 \\ & + B_1\beta_1 + B_2\beta_2 + B_3\beta_3 \\ & + C_1\gamma_1 + C_2\gamma_2 + C_3\gamma_3.\end{aligned}$$

$$\phi_2 = (a_1^2)\, \alpha_1^2 + (a_2^2)\, \alpha_2^2 + \&c.$$
$$+ (a_1 b_2) \cdot \alpha_1\beta_2 + \&c., \quad + (c_3 a_2) \cdot \gamma_3\alpha_2 + \&c.;$$

the expressions within the brackets denoting the corresponding coefficients.

$\phi_1$ will contain nine coefficients, which will be constant, or functions of $(x, y, z)$, according as there are no external forces, or *vice versâ;* and $\phi_2$ will contain forty-five distinct coefficients. I shall consider these functions separately. Introducing $\phi_1$ for $V$ in equation (4), we obtain,

$$\begin{aligned}\iiint \epsilon \left(\frac{d^2\xi}{dt^2}\,\delta\xi + \frac{d^2\eta}{dt^2}\,\delta\eta + \frac{d^2\zeta}{dt^2}\,\delta\zeta\right) dxdydz = {} & \iint (A_1\delta\xi + B_1\delta\eta + C_1\delta\zeta)\, dydz, \\ & + \iint (A_2\delta\xi + B_2\delta\eta + C_2\delta\zeta)\, dxdz, \qquad (12) \\ & + \iint (A_3\delta\xi + B_3\delta\eta + C_3\delta\zeta)\, dxdy\end{aligned}$$

$$-\iiint\left\{\left(\frac{dA_1}{dx}+\frac{dA_2}{dy}+\frac{dA_3}{dz}\right)\delta\xi+\left(\frac{dB_1}{dx}+\frac{dB_2}{dy}+\frac{dB_3}{dz}\right)\delta\eta+\left(\frac{dC_1}{dx}+\frac{dC_2}{dy}+\frac{dC_3}{dz}\right)\delta\zeta\right\}dxdydz,$$

whence we obtain

$$\begin{aligned} -\epsilon\frac{d^2\xi}{dt^2} &= \frac{dA_1}{dx} + \frac{dA_2}{dy} + \frac{dA_3}{dz}, \\ -\epsilon\frac{d^2\eta}{dt^2} &= \frac{dB_1}{dx} + \frac{dB_2}{dy} + \frac{dB_3}{dz}, \qquad (13) \\ -\epsilon\frac{d^2\zeta}{dt^2} &= \frac{dC_1}{dx} + \frac{dC_2}{dy} + \frac{dC_3}{dz}.\end{aligned}$$

The terms on the right hand side of equation (13) must be added to the dynamical equations arising from $\phi_2$, even in cases where $(X, Y, Z)$ may be neglected on account of the intensity of the molecular forces. If we wish to take account of the external forces, we should use, instead of (13), the following:

$$-\epsilon\frac{d^2\xi}{dt^2}=\frac{dA_1}{dx}+\frac{dA_2}{dy}+\frac{dA_3}{dz}-\epsilon X;$$

$$-\epsilon\frac{d^2\eta}{dt^2}=\frac{dB_1}{dx}+\frac{dB_2}{dy}+\frac{dB_3}{dz}-\epsilon Y;$$

$$-\epsilon\frac{d^2\zeta}{dt^2}=\frac{dC_1}{dx}+\frac{dC_2}{dy}+\frac{dC_3}{dz}-\epsilon Z.$$

If no external forces act, equations (13) disappear, in consequence of $A_1$, $A_2$, &c., becoming constants; but the conditions at the limits (12) will still remain, unless the coefficients be not only constant, but zero.

The part of the differential equations of motion depending on $\phi_2$ will be found by introducing $\delta\phi_2$ in place of $\delta V$ in equation (3), and integrating by parts, according to the methods of the calculus of variations; but it may be found more readily by introducing $\phi_2$ for $V$, in equation (4). Neglecting the equations of condition at the limits, we obtain for the equations of motion,

$$\begin{aligned}
-\epsilon\frac{d^2\xi}{dt^2}&=(a_1^2)\frac{d^2\xi}{dx^2}+(a_2^2)\frac{d^2\xi}{dy^2}+(a_3^2)\frac{d^2\xi}{dz^2}\\
&+2(a_2a_3)\frac{d^2\xi}{dydz}+2(a_1a_3)\frac{d^2\xi}{dxdz}+2(a_1a_2)\frac{d^2\xi}{dxdy}\\
&+(a_1b_1)\frac{d^2\eta}{dx^2}+(a_2b_2)\frac{d^2\eta}{dy^2}+(a_3b_3)\frac{d^2\eta}{dz^2}\\
&+(a_2b_3+a_3b_2)\frac{d^2\eta}{dydz}+(a_1b_3+a_3b_1)\frac{d^2\eta}{dxdz}+(a_1b_2+a_2b_1)\frac{d^2\eta}{dxdy}\\
&+(a_1c_1)\frac{d^2\zeta}{dx^2}+(a_2c_2)\frac{d^2\zeta}{dy^2}+(a_3c_3)\frac{d^2\zeta}{dz^2}\\
&+(a_2c_3+a_3c_2)\frac{d^2\zeta}{dydz}+(a_1c_3+a_3c_1)\frac{d^2\zeta}{dxdz}+(a_1c_2+a_2c_1)\frac{d^2\zeta}{dxdy}.
\end{aligned}\tag{14}$$

$$\begin{aligned}
-\epsilon\frac{d^2\eta}{dt^2}&=(b_1^2)\frac{d^2\eta}{dx^2}+(b_2^2)\frac{d^2\eta}{dy^2}+(b_3^2)\frac{d^2\eta}{dz^2}\\
&+2(b_2b_3)\frac{d^2\eta}{dydz}+2(b_1b_3)\frac{d^2\eta}{dxdz}+2(b_1b_2)\frac{d^2\eta}{dxdy}
\end{aligned}$$

$$+ (b_1c_1)\frac{d^2\zeta}{dx^2} + (b_2c_2)\frac{d^2\zeta}{dy^2} + (b_3c_3)\frac{d^2\zeta}{dz^2}$$

$$+ (b_2c_3 + b_3c_2)\frac{d^2\zeta}{dydz} + (b_1c_3 + b_3c_1)\frac{d^2\zeta}{dxdz} + (b_1c_2 + b_2c_1)\frac{d^2\zeta}{dxdy}$$

$$+ (a_1b_1)\frac{d^2\xi}{dx^2} + (a_2b_2)\frac{d^2\xi}{dy^2} + (a_3b_3)\frac{d^2\xi}{dz^2}$$

$$+(a_2b_3+a_3b_2)\frac{d^2\xi}{dydz} + (a_1b_3+a_3b_1)\frac{d^2\xi}{dxdz} + (a_1b_2+a_2b_1)\frac{d^2\xi}{dxdy}.$$

$$-\epsilon\frac{d^2\zeta}{dt^2} = (c_1^2)\frac{d^2\zeta}{dx^2} + (c_2^2)\frac{d^2\zeta}{dy^2} + (c_3^2)\frac{d^2\zeta}{dz^2}$$

$$+2(c_2c_3)\frac{d^2\zeta}{dydz} + 2(c_1c_3)\frac{d^2\zeta}{dxdz} + 2(c_1c_2)\frac{d^2\zeta}{dxdy}$$

$$+ (a_1c_1)\frac{d^2\xi}{dx^2} + (a_2c_2)\frac{d^2\xi}{dy^2} + (a_3c_3)\frac{d^2\xi}{dz^2}$$

$$+ (a_2c_3 + a_3c_2)\frac{d^2\xi}{dydz} + (a_1c_3 + a_3c_1)\frac{d^2\xi}{dxdz} + (a_1c_2 + a_2c_1)\frac{d^2\xi}{dxdy}$$

$$+ (b_1c_1)\frac{d^2\eta}{dx^2} + (b_2c_2)\frac{d^2\eta}{dy^2} + (b_3c_3)\frac{d^2\eta}{dz^2}$$

$$+ (b_2c_3 + b_3c_2)\frac{d^2\eta}{dydz} + (b_1c_3+b_3c_1)\frac{d^2\eta}{dxdz} + (b_1c_2+b_2c_1)\frac{d^2\eta}{dxdy}.$$

By combining these equations with (12), and comparing (4), we obtain from the function $V = \phi_1 + \phi_2$ the following equations of motion,

$$\begin{aligned}
-\epsilon\frac{d^2\xi}{dt^2} &= \frac{d}{dx}\left(\frac{d\phi_2}{da_1}+A_1\right) + \frac{d}{dy}\left(\frac{d\phi_2}{da_2}+A_2\right) + \frac{d}{dz}\left(\frac{d\phi_2}{da_3}+A_3\right) \\
-\epsilon\frac{d^2\eta}{dt^2} &+ \frac{d}{dx}\left(\frac{d\phi_2}{d\beta_1}+B_1\right) + \frac{d}{dy}\left(\frac{d\phi_2}{d\beta_2}+B_2\right) + \frac{d}{dz}\left(\frac{d\phi_2}{d\beta_3}+B_3\right) \qquad (15) \\
-\epsilon\frac{d^2\zeta}{dt^2} &= \frac{d}{dx}\left(\frac{d\phi_2}{d\gamma_1}+C_1\right) + \frac{d}{dy}\left(\frac{d\phi_2}{d\gamma_2}+C_2\right) + \frac{d}{dz}\left(\frac{d\phi_2}{d\gamma_3}+C_3\right)
\end{aligned}$$

These are the equations of small oscillations of all media, for which the direct influence of the external forces may be neglected.

In equations (14) it will be observed that nine of the coefficients are

composed of two terms, coefficients of the original function $\phi_2$; these compound coefficients will, in the equations of motion, be equivalent to simple coefficients, so that the total number of distinct constants in the equations of wave-motion will be reduced to thirty-six; this reduction cannot, however, be introduced into the conditions at the limits, because each coefficient will at the limits be multiplied by a different function of $(x, y, z)$. Hence may be deduced an important theorem, which I shall prove in a general manner.

" Two bodies, different in their molecular structure, may have the same laws of wave-propagation, but cannot have the same laws of reflexion and refraction."

The laws of wave-propagation (5) depend upon the differential coefficients of $\left(\frac{dV}{da_1}, \&c.\right)$, taken with respect to $(x, y, z)$; but the laws of reflexion and refraction (6) depend on the quantities $\left(\frac{dV}{da_1}, \&c.\right)$ themselves. Each of these quantities is of the form

$$A\frac{d\xi}{dx} + B\frac{d\xi}{dy} + C\frac{d\xi}{dz} + D\frac{d\eta}{dx} + E\frac{d\eta}{dy} + F\frac{d\eta}{dz} + G\frac{d\zeta}{dx} + H\frac{d\zeta}{dy} + I\frac{d\zeta}{dz}.$$

If, therefore, for example, in the first of the equations (5), there be in $\left(\frac{dV}{da_1}\right)$ a term of the form $\left(B\frac{d\xi}{dy}\right)$, there will be in $\frac{dV}{da_2}$ a term of the form $\left(A'\frac{d\xi}{dx}\right)$; then, differentiating the first with respect to $x$, and the second with respect to $y$, we shall obtain, as part of the equations of motion, $(B + A')\frac{d^2\xi}{dxdy}$, while the corresponding part of the conditions at the limits will be $\iint B\frac{d\xi}{dy}\delta\xi dydz + \iint A'\frac{d\xi}{dx}\delta\xi dxdz$. If, now, we suppose another body such that $\left(A'\frac{d\xi}{dy}\right)$ and $\left(B\frac{d\xi}{dx}\right)$ are terms in $\left(\frac{dV}{da_1}\right)$ and $\left(\frac{dV}{da_2}\right)$, the part resulting from these quantities in the equations of motion will be $(A' + B)\frac{d^2\xi}{dxdy}$, which is identical with the former value, while the corresponding portion of the conditions at the

limits will be $\iint A' \frac{d\xi}{dy} \delta\xi dydz + \iint B \frac{d\xi}{dx} \delta\xi dxdz$ ; which cannot be reduced to the former value, unless $A' = B$.

It is sufficient here to establish the general theorem. I shall return to it again, in comparing Professor MAC CULLAGH's theory of light and that advocated by M. CAUCHY and Mr. GREEN, which give the same laws of wave-propagation, but differ in the laws of reflexion and refraction.

I shall now integrate the equations of motion (14), for the particular case of plane waves and rectilinear vibrations. Let $(l, m, n)$ be the direction cosines of the wave-normal, $(\alpha, \beta, \gamma)$ the direction of molecular vibration, and $(v)$ the velocity of the wave ; then the integral will be

$$\xi = \cos \alpha \cdot f \left\{ \frac{2\pi}{\lambda} (lx + my + nz - vt) \right\},$$

$$\eta = \cos \beta \cdot f \left\{ \frac{2\pi}{\lambda} (lx + my + nz - vt) \right\},$$

$$\zeta = \cos \gamma \cdot f \left\{ \frac{2\pi}{\lambda} (lx + my + nz - vt) \right\}.$$

Introducing these values into the equations (14), we obtain

$$\begin{aligned} \epsilon v^2 \cos \alpha &= P' \cos \alpha + H' \cos \beta + G' \cos \gamma, \\ \epsilon v^2 \cos \beta &= Q' \cos \beta + F' \cos \gamma + H' \cos \alpha, \\ \epsilon v^2 \cos \gamma &= R' \cos \gamma + G' \cos \alpha + F' \cos \beta ; \end{aligned} \tag{16}$$

where

$$P' = (a_1^2)\, l^2 + (a_2^2)\, m^2 + (a_3^2)\, n^2 + 2(a_2 a_3)\, mn + 2(a_1 a_3)\, ln + 2(a_1 a_2)\, lm ;$$
$$Q' = (b_1^2)\, l^2 + (b_2^2)\, m^2 + (b_3^2)\, n^2 + 2(b_2 b_3)\, mn + 2(b_1 b_3)\, ln + 2(b_1 b_2)\, lm ;$$
$$R' = (c_1^2)\, l^2 + (c_2^2)\, m^2 + (c_3^2)\, n^2 + 2(c_2 c_3)\, mn + 2(c_1 c_3)\, ln + 2(c_1 c_2)\, lm ;$$
$$F' = (b_1 c_1)\, l^2 + (b_2 c_2)\, m^2 + (b_3 c_3)\, n^2 + (b_2 c_3 + b_3 c_2)\, mn + (b_1 c_3 + b_3 c_1)\, ln + (b_1 c_2 + b_2 c_1)\, lm ;$$
$$G' = (a_1 c_1)\, l^2 + (a_2 c_2)\, m^2 + (a_3 c_3)\, n^2 + (a_2 c_3 + a_3 c_2)\, mn + (a_1 c_3 + a_3 c_1)\, ln + (a_1 c_2 + a_2 c_1)\, lm ;$$
$$H' = (a_1 b_1)\, l^2 + (a_2 b_2)\, m^2 + (a_3 b_3)\, n^2 + (a_2 b_3 + a_3 b_2)\, mn + (a_1 b_3 + a_3 b_1)\, ln + (a_1 b_2 + a_2 b_1)\, lm.$$

Equations (16) are the well-known equations for determining the axes of the ellipsoid whose equation is

$$P'x^2 + Q'y^2 + R'z^2 + 2F'yz + 2G'xz + 2H'xy = 1. \tag{17}$$

There are, therefore, three possible directions of molecular vibration for a given direction of wave plane; and there will be three parallel waves moving with velocities determined by the magnitude of the axes of the ellipsoid; the direction of vibration in each wave being parallel to one of the axes. The theorem involved in equation (16) was first proved by M. CAUCHY,* for bodies whose molecules act by attractions and repulsions in the line joining them. It is here extended to every kind of molecular action, and shown to be a fundamental property in molecular dynamics. It may be worth while to examine the reason of its truth. It arises from the homogeneity of equations (14) resulting from the absence of external forces. The right hand member of each of the three equations of motion consists of eighteen terms; there will, therefore, be in all fifty-four terms; if each of these were supposed to have different and independent coefficients, equations (16) would cease to be true, and the coefficients of ($\cos \alpha$, $\cos \beta$, $\cos \gamma$) would be nine distinct quantities, so that the theorem which represents the direction and magnitude of the molecular vibration by means of an ellipsoid whose coefficients are functions of the direction of the wave, would no longer be applicable. Equations (14) contain only thirty-six distinct constants, and this reduction in the number of constants arises from the assumption that the virtual moments of the system may be represented by $\iiint \delta V dxdydz$; which is equivalent to assuming that the virtual moments of the molecular forces applied at any point may be represented by the variation of a single function:

$$Q\delta q + Q'\delta q' + \&c. = \delta V.$$

The cubic equation whose roots are the squares of the reciprocals of the axes of the ellipsoid (17) is

$$(P'-s)(Q'-s)(R'-s)-F'^2(P'-s)-G'^2(Q'-s)-H'^2(R'-s)+2F'G'H'=0;$$

where $s=\epsilon v^2$. Hence, if ($P, Q, R, F, G, H$) denote the same functions of ($x, y, z$) that ($P', Q', R', F', G', H'$) are of ($l, m, n$), it may be shown, in a manner similar

* Exercices de Mathematiques, tom. v. p. 32.

to the method for bodies whose molecules act in the line joining them, that the surface of *wave-slowness** will be

$$(P-1)(Q-1)(R-1)-F^2(P-1)-G^2(Q-1)-H^2(R-1)+2FGH=0. \quad (18)$$

$(P-1, Q-1, \&c.)$ equated to zero, denoting the six fixed ellipsoids used in the memoir referred to; these ellipsoids thus appear in the general theory of molecular dynamics, and will have the same use in interpreting the conditions at the limits in the general problem, as in the particular case of attracting and repelling molecules. As their use has been fully explained in my former memoir, I shall only refer to it, and pass on to other subjects.

I shall here state the general conditions necessary, in order that the molecular equilibrium of a body removed from the influence of external forces should be stable; this investigation will be found of use in the subsequent part of this memoir. The equation of virtual velocities, in the case supposed, will become

$$\iiint \delta V dxdydz = 0;$$

or, as it may be more accurately stated,

$$\iiint \delta V dxdydz = 0, \text{ or } < 0;$$

this sum never becoming positive for possible displacements. The equation of virtual velocities, as stated by M. POISSON and other writers, supposes no virtual displacements but those for which equal and opposite displacements are possible; the correct statement of the principle is, that the sum of the virtual moments of the system can never become positive for possible displacements. For the full development of this important correction of the equation of virtual velocities, as given by LAGRANGE, I shall refer to a memoir of M. OSTROGRADSKY, contained in the *Memoires de l'Academie de St. Petersbourg.*†

The application of the principle to the present case will be evident upon the statement of the question. As the form of $V$ is given in terms of $(a_1, a_2, \&c.)$, we shall have

$$\delta V = \frac{dV}{da_1}\delta a_1 + \frac{dV}{da_2}\delta a_2 + \&c.;$$

* *Vide* Transactions of the Royal Irish Academy, vol. xxi. part ii. p. 172.

† Tom. iii. p. 130.

$\left(\frac{dV}{da_1}, \frac{dV}{da_2}, \text{\&c.}\right)$ denoting forces tending to alter the quantities $(a_1, a_2, a_3, \text{\&c.})$, which are functions of $(x, y, z)$. The whole body may, therefore, be divided into *couches* by a series of surfaces whose equation will be

$$a_1 - C = 0;$$

$C$ denoting the parameter of the system. In a similar manner, the body may be conceived as divided into *couches* by other sets of surfaces corresponding to $(a_2, a_3, \text{\&c.})$. In any one of these *couches* the corresponding function $(a_1, a_2, \text{\&c.})$ will have a constant value, which will vary from one *couche* to another.

If $L = 0$ be the equation of a surface, we may conceive all space as divided into two portions, which will be distinguished by the property, that the portion lying at one side of the surface will have the function of $(x, y, z)$ denoted by $(L)$, positive; while for the rest of space, lying at the other side of the surface, the function $(L)$ will be negative. Similarly, $\delta L$ will be positive for all displacements made at one side of the surface; negative for all displacements on the opposite side; and zero for displacements in the surface itself.

Let us now resume the equation, $a_1 - C = 0$, which denotes a surface drawn in the interior of the body, along which the value of $\frac{d\xi}{dx}$ remains constant. It is necessary for the stable equilibrium of the body, that if the particles composing this surface be displaced from it, the molecular forces developed by the displacement should tend to restore them to the surface; this condition will require that $\left(\frac{dV}{da_1}, \frac{dV}{da_2}, \text{\&c.}\right)$, which are the forces developed by the displacements $(\delta a_1, \delta a_2, \text{\&c.})$, should have signs opposite to those of the displacements. If no forces are developed tending to restore the molecules, we shall have

$$\frac{dV}{da_1} = 0, \quad \frac{dV}{da_2} = 0, \text{ \&c.};$$

and these equations will determine the *limits* of stable and unstable equilibrium in the body.

## SECTION II.—LAWS OF NORMAL VIBRATIONS.

It has been shown that the differential coefficients of the function $V$, taken with respect to $(a_1, a_2, \&c.)$, denote the normal and tangential actions of the surrounding body on an elementary parallelepiped. I shall now suppose that the body is so constituted that the tangential forces vanish, and so deduce the laws of a body capable of transmitting pressure in a normal direction only. The condition that the tangential forces vanish will give the equations,

$$\frac{dV}{da_2} = 0, \quad \frac{dV}{d\beta_1} = 0, \quad \frac{dV}{d\gamma_1} = 0,$$
$$\frac{dV}{da_3} = 0, \quad \frac{dV}{d\beta_3} = 0, \quad \frac{dV}{d\gamma_2} = 0,$$

which will reduce the function to the form

$$2V = Aa_1^2 + B\beta_2^2 + C\gamma_3^2 + 2P\beta_2\gamma_3 + 2Qa_1\gamma_3 + 2Ra_1\beta_2.$$

At first sight, it might appear that we might assume this or any other form of function to define a body, and proceed to deduce the laws of motion; but, as it is possible that an assumed form of the function may be only true for particular axes of co-ordinates, it is always necessary to examine whether a transformation of co-ordinates will introduce any new terms; if this be the case, then the assumed function will not represent completely the molecular structure of the body, as it will be merely a simplification of a more complex function, produced by assuming particular axes of co-ordinates, which are connected with the crystalline structure of the body. I shall examine the function just given for normal pressures by this method, and determine it so as to be independent of the axes of co-ordinates. The relations between the two systems of variables are

$$x = ax' + by' + cz',$$
$$y = a'x' + b'y' + c'z',$$
$$z = a''x' + b''y' + c''z';$$

also $(\xi, \eta, \zeta)$ will satisfy these equations. Let us now assume

$$u = \beta_3 + \gamma_2, \quad v = \gamma_1 + a_3, \quad w = a_2 + \beta_1,$$

we may immediately deduce, by the formulæ for transformation of the independent variables, the following values for the nine quantities ($\alpha_1$, $\alpha_2$, &c.),

$$\begin{aligned}
\alpha_1 &= a^2\alpha_1' + b^2\beta_2' + c^2\gamma_3' + bcu' + acv' + abw', \\
\beta_2 &= a'^2\alpha_1' + b'^2\beta_2' + c'^2\gamma_3' + b'c'u' + a'c'v' + a'b'w', \\
\gamma_3 &= a''^2\alpha_1' + b''^2\beta_2' + c''^2\gamma_3' + b''c''u' + a''c''v' + a''b''w'.
\end{aligned}$$

$$\beta_3 = \left\{\begin{array}{l} \quad a'\ (a''\alpha_1' + b''\alpha_2' + c''\alpha_3') \\ + b'\ (a''\beta_1' + b''\beta_2' + c''\beta_3') \\ + c'\ (a''\gamma_1' + b''\gamma_2' + c''\gamma_3') \end{array}\right\} \quad \gamma_2 = \left\{\begin{array}{l} \quad a''\ (a'\alpha_1' + b'\alpha_2' + c'\alpha_3') \\ + b''\ (a'\beta_1' + b'\beta_2' + c'\beta_3') \\ + c''\ (a'\gamma_1' + b'\gamma_2' + c'\gamma_3') \end{array}\right\}$$

$$\gamma_1 = \left\{\begin{array}{l} \quad a''\ (a\alpha_1' + b\alpha_2' + c\alpha_3') \\ + b''\ (a\beta_1' + b\beta_2' + c\beta_3') \\ + c''\ (a\gamma_1' + b\gamma_2' + c\gamma_3') \end{array}\right\} \quad \alpha_3 = \left\{\begin{array}{l} \quad a\ (a''\alpha_1' + b''\alpha_2' + c''\alpha_3') \\ + b\ (a''\beta_1' + b''\beta_2' + c''\beta_3') \\ + c\ (a''\gamma_1' + b''\gamma_2' + c''\gamma_3') \end{array}\right\} \qquad (19)$$

$$\alpha_2 = \left\{\begin{array}{l} \quad a\ (a'\alpha_1' + b'\alpha_2' + c'\alpha_3') \\ + b\ (a'\beta_1' + b'\alpha_2' + c'\beta_3') \\ + c\ (a'\gamma_1' + b'\gamma_2' + c'\gamma_3') \end{array}\right\} \quad \beta_1 = \left\{\begin{array}{l} \quad a'\ (a\alpha_1' + b\alpha_2' + c\alpha_3') \\ + b'\ (a\beta_1' + b\beta_2' + c\beta_3') \\ + c'\ (a\gamma_1' + b\gamma_2' + c\gamma_3') \end{array}\right\}$$

It is plain from the first three equations that the change of the co-ordinate axes will introduce into the function $V$ three new variables $(u, v, w)$. Hence, the function which I have assumed is only admissible for particular axes, and the true form of the function from which it is derived will be that of a function of the six variables used in my former memoir.

It is, however, possible to obtain such a function of $\left(\frac{d\xi}{dx}, \frac{d\eta}{dy}, \frac{d\zeta}{dz}\right)$ as shall not change its form with the transformation of co-ordinates. For, adding together the values of these quantities, we shall banish $(u', v', w')$, on account of the relations,

$$\begin{aligned}
bc + b'c' + b''c'' &= 0, \\
ac + a'c' + a''c'' &= 0, \\
ab + a'b' + a''b'' &= 0.
\end{aligned}$$

The result of the addition will be

$$\frac{d\xi}{dx} + \frac{d\eta}{dy} + \frac{d\zeta}{dz} = \frac{d\xi'}{dx'} + \frac{d\eta'}{dy'} + \frac{d\zeta'}{dz'} = \omega.$$

This quantity ($\omega$) will retain its value, independent of the directions of the

co-ordinates, and consequently the proper form of the function $V$, for bodies which transmit only normal pressure in every direction, will be

$$V = F(\varpi). \tag{20}$$

There are two other forms of the function $V$ (considered as containing the nine differential coefficients) which possess this property of reproducing themselves by transformation of co-ordinates. Let $(X, Y, Z)$ be determined by the following equations,

$$X = \beta_3 - \gamma_2, \quad Y = \gamma_1 - \alpha_3, \quad Z = \alpha_2 - \beta_1.$$

It appears immediately from equations (19) that $(X, Y, Z)$ are expressed by the following relations in terms of $(X', Y', Z')$,

$$\begin{aligned} X &= aX' + bY' + cZ', \\ Y &= a'X' + b'Y' + c'Z', \\ Z &= a''X' + b''Y' + c''Z'. \end{aligned}$$

These formulæ are well known, and prove that another self-producing form of the function $V$ will be

$$V = F(X, Y, Z). \tag{21}$$

I have proved in my former memoir that if $(\alpha, \beta, \gamma, u, v, w)$ denote the coefficients $\left\{\frac{d\xi}{dx}, \frac{d\eta}{dy}, \frac{d\zeta}{dz}, \left(\frac{d\eta}{dz} + \frac{d\zeta}{dy}\right), \left(\frac{d\zeta}{dx} + \frac{d\xi}{dz}\right), \left(\frac{d\xi}{dy} + \frac{d\eta}{dx}\right)\right\}$, they will become, by transformation of co-ordinates, linear functions of $(\alpha', \beta', \gamma', u', v', w')$; and that the formulæ of tranformation are the same as for $(x^2)$, $(y^2)$, $(z^2)$, $(2yz)$, $(2xz)$, $(2xy)$. Hence, a function of these six quantities will reproduce itself, by transformation of axes of co-ordinates. The third form of $V$ which possesses this property is, therefore,

$$V = F(\alpha, \beta, \gamma, u, v, w). \tag{22}$$

The forms of the function $V$, which have just been determined, are perfectly general, and do not merely express properties of the body with reference to particular axes, but general properties of its molecular structure, independent of the directions of the co-ordinates.

I shall consider in the present section the function (20), which is peculiar

to bodies which transmit normal pressure in every direction. Before deducing the equations of motion, it may be useful to show, in another manner, the necessity for reducing the function of $(\alpha, \beta, \gamma)$ to a function of $(\alpha + \beta + \gamma)$. By the theorem proved in my former memoir, $(\alpha, \beta, \gamma)$ are transformed by the same equations as $(x^2, y^2, z^2)$ ; hence, $V = F(\alpha, \beta, \gamma)$ may be represented by a surface whose equation contains only the squares of the co-ordinates. It is evident that such a surface will be symmetrical with respect to the co-ordinate planes ; and if this condition be satisfied for every system of co-ordinates, the surface must be a sphere ; hence,

$$V = F(\alpha, \beta, \gamma) = F(\alpha + \beta + \gamma).$$

The equations of motion deduced from (20) will be the equations commonly used in hydrodynamics, and for this reason it may be useful to state them in their most general form. Let $(x, y, z)$ denote, as in equations (1) and (2), the actual positions of the molecules ; then, since

$$\delta V = F\left(\frac{d\delta x}{dx} + \frac{d\delta y}{dy} + \frac{d\delta z}{dz}\right),$$

we shall obtain from equations (2)

$$\begin{aligned}
\frac{1}{\rho}\frac{dp}{dx} &= X - \frac{du}{dt} - u\frac{du}{dx} - v\frac{du}{dy} - w\frac{du}{dz}\,;\\
\frac{1}{\rho}\frac{dp}{dy} &= Y - \frac{dv}{dt} - u\frac{dv}{dx} - v\frac{dv}{dy} - w\frac{dv}{dz}\,;\\
\frac{1}{\rho}\frac{dp}{dz} &= Z - \frac{dw}{dt} - u\frac{dw}{dx} - v\frac{dw}{dy} - w\frac{dw}{dz}\,;
\end{aligned} \qquad (23,\ \text{a})$$

assuming $p = \frac{dV}{d\omega}$, and recollecting that $(u', v', w')$ are the *total* differential coefficients of $(u, v, w)$, taken with respect to $(t)$. These are the mechanical equations of the problem, and, combined with the equation of continuity,

$$\frac{d\rho}{dt} + \frac{d(\rho u)}{dx} + \frac{d(\rho v)}{dy} + \frac{d(\rho w)}{dz} = 0, \qquad (24,\ \text{a})$$

contain all that is requisite to determine the motion of points situated in the interior of the mass. The mechanical conditions to be satisfied at the limits are contained in the double integrals, which become, by the substitution of the element of the bounding surface,

$$\iint p dS\,(\delta x \cos\lambda + \delta y \cos\mu + \delta z \cos\nu),$$

$(\lambda, \mu, \nu)$ denoting the direction of the normal. It may be easily shown from this expression, that if $P$ denote a function of $(x, y, z, t)$, expressing the normal forces applied at each point of the surface, the mechanical condition at the limits will be expressed by the equation,

$$p - P = 0. \qquad (23,\ \mathrm{b})$$

To which must be added the equation of the bounding surface, deduced from geometrical considerations,

$$\frac{dF}{dt} + u\frac{dF}{dx} + v\frac{dF}{dy} + w\frac{dF}{dz} = 0. \qquad (24,\ \mathrm{b})$$

It is unnecessary to enter further into this subject, as it is fully treated by LAGRANGE, in the second volume of the *Mecanique Analytique*, and is indeed the only example given by him of the application of his formulæ to the problem of bodies composed of continuous points.

Resuming the former signification of $(x, y, z, \xi, \eta, \zeta)$, we find from equations (5) or (15), since $2V = 2p\omega + A\omega^2$, and $\frac{dV}{d\omega} = p + A\omega$,

$$-\epsilon\frac{d^2\xi}{dt^2} = \frac{dp}{dx} + A\frac{d\omega}{dx},$$

$$-\epsilon\frac{d^2\eta}{dt^2} = \frac{dp}{dy} + A\frac{d\omega}{dy},$$

$$-\epsilon\frac{d^2\zeta}{dt^2} = \frac{dp}{dz} + A\frac{d\omega}{dz}.$$

If no external forces act, $p$ and $A$ will be constants; and, in this case, these equations become, assuming a negative sign for $A$, so that the equilibrium may be stable,

$$\begin{aligned}
\epsilon \frac{d^2\xi}{dt^2} &= A \frac{d\omega}{dx}, \\
\epsilon \frac{d^2\eta}{dt^2} &= A \frac{d\omega}{dy}, \\
\epsilon \frac{d^2\zeta}{dt^2} &= A \frac{d\omega}{dz}.
\end{aligned} \tag{25}$$

and, by differentiating with respect to $(x, y, z)$, we obtain

$$\frac{d^2\omega}{dt^2} = \frac{A}{\epsilon}\left(\frac{d^2\omega}{dx^2} + \frac{d^2\omega}{dy^2} + \frac{d^2\omega}{dz^2}\right), \tag{26}$$

which determines the cubical compression as a function of $(x, y, z, t)$.

The equations (25) and (26) are applicable to the propagation of sound in gases, and to the longitudinal vibrations of elastic solids; but there will be an essential difference between the two cases. If the gas were freed from the action of external forces, and unconfined at its limits, its molecules would separate, and a displacement would develope no force tending to restore them to their original positions; hence, $\left(\frac{dV}{d\omega}\delta\omega\right)$ would become positive, which is inconsistent with stable equilibrium, and the velocity of wave propagation $\left(\sqrt{\frac{A}{\epsilon}}\right)$ would become imaginary. In order, therefore, that the equilibrium of the gas be stable, we must suppose a constant pressure exerted at the limits, sufficient to keep the molecules together, and restore them to their original positions, if displaced. No such condition is requisite in the elastic solid, for $\left(\frac{dV}{d\omega}\delta\omega\right)$ will be negative, without the assistance of forces applied at the bounding surface.

## SECTION III. LAWS OF TRANSVERSE VIBRATIONS.

In order to obtain the form of the function $V$ peculiar to transverse vibrations, we must suppose that, if a plane be drawn through the body in any direction, the molecular forces exerted on any element of the plane will be altogether tangential. Hence we obtain

R 2

$$\frac{dV}{d\alpha_1} = 0, \quad \frac{dV}{d\beta_2} = 0, \quad \frac{dV}{d\gamma_3} = 0;$$

and the function $V$ will become

$$V = F(\alpha_2, \alpha_3, \beta_1, \beta_3, \gamma_1, \gamma_2).$$

But, as I have already shown, such a form must be assumed, as will reproduce itself by transformation of co-ordinates. The only function which satisfies this condition is the function (21),

$$V = F(X, Y, Z).$$

This function, deduced in a different manner, has been used by Professor MAC CULLAGH in his mechanical theory of light; and for the discussion of its properties and the laws of propagation, reflection, and refraction, deduced from it, I shall refer to his memoir in the Transactions of the Royal Irish Academy.* As, however, I shall have occasion to use them hereafter, I shall here state the differential equations of motion, and the conditions at the limits. On account of the form of the function, the following relations exist:

$$\frac{dV}{dX} = \frac{dV}{d\beta_3} = -\frac{dV}{d\gamma_2};$$
$$\frac{dV}{dY} = \frac{dV}{d\gamma_1} = -\frac{dV}{d\alpha_3};$$
$$\frac{dV}{dZ} = \frac{dV}{d\alpha_2} = -\frac{dV}{d\beta_1}.$$

Hence equations (5) will become

$$\begin{aligned} -\epsilon\frac{d^2\xi}{dt^2} &= \frac{d}{dy}\cdot\frac{dV}{dZ} - \frac{d}{dz}\cdot\frac{dV}{dY}; \\ -\epsilon\frac{d^2\eta}{dt^2} &= \frac{d}{dz}\cdot\frac{dV}{dX} - \frac{d}{dx}\cdot\frac{dV}{dZ}; \\ -\epsilon\frac{d^2\zeta}{dt^2} &= \frac{d}{dx}\cdot\frac{dV}{dY} - \frac{d}{dy}\cdot\frac{dV}{dX}. \end{aligned} \qquad (27)$$

* Vol. xxi. p. 17.

These are the equations of motion; also equations (6) will become

$$\left(\frac{dV'_0}{dZ}-\frac{dV''_0}{dZ}\right)\frac{dF}{dy}-\left(\frac{dV'_0}{dY}-\frac{dV''_0}{dY}\right)\frac{dF}{dz}=0;$$
$$\left(\frac{dV'_0}{dX}-\frac{dV''_0}{dX}\right)\frac{dF}{dz}-\left(\frac{dV'_0}{dZ}-\frac{dV''_0}{dZ}\right)\frac{dF}{dx}=0; \qquad (28)$$
$$\left(\frac{dV'_0}{dY}-\frac{dV''_0}{dY}\right)\frac{dF}{dx}-\left(\frac{dV'_0}{dX}-\frac{dV''_0}{dX}\right)\frac{dF}{dy}=0.$$

These are the conditions to be fulfilled at the limiting surface $F(x,y,z)=0$. They are equivalent to two conditions only, as the third equation may be deduced from the first two. The reason of this is evident *à priori;* for the conditions at the limits express in general, that the normal and two tangential pressures arising from the molecular forces in each body equilibrate each other for every point of the separating surface; but in the present case there are no normal pressures; hence there can be only two mechanical conditions at the limits. To these conditions at the limits, arising from the mechanical equations, should be added the three geometrical conditions resulting from the equivalence of vibrations; these conditions are

$$\xi'_0=\xi''_0, \quad \eta'_0=\eta''_0, \quad \zeta'_0=\zeta''_0.$$

These, together with the mechanical conditions, will be equivalent to five equations, which Professor MAC CULLAGH reduces to four by the hypothesis that the density of the luminiferous medium is the same in all transparent bodies; this hypothesis is necessary in order to reduce the number of equations to the number of unknown quantities in the problem.

If no external forces act upon the system, the function $V$ will be reduced to a homogeneous function of the second order,

$$2V=PX^2+QY^2+RZ^2+2FYZ+2GXZ+2HXY;$$

and since $(X, Y, Z)$ are transformed by a change of co-ordinates, in the same manner as $(x, y, z)$, it is evident that the coefficients of the rectangles will vanish for axes of co-ordinates which coincide with the axes of the ellipsoid,

$$Px^2+Qy^2+Rz^2+2Fyz+2Gxz+2Hxy=1.$$

Hence the simplest form of the function will be

$$2V = PX^2 + QY^2 + RZ^2.$$

The axes of co-ordinates are in this formula the axes of elasticity.

### SECTION IV. EQUATIONS OF A SYSTEM WHOSE MOLECULES ATTRACT AND REPEL EACH OTHER.

The third form of the function $V$, which possesses the property of reproducing itself by a transformation of co-ordinates, is given by equation (22),

$$V = F(\alpha, \beta, \gamma, u, v, w).$$

The six variables contained in this function are the quantities upon which a change in the distance between the molecules depends; hence the variation of this function will express the virtual moments of a system of forces tending to alter this distance. In my former communication* I have made use of this function, using definite integrals to represent the coefficients; the equations thus found contain a smaller number of constants, than if the coefficients had been assumed to be arbitrary, without any relation expressed by definite integrals. I shall here use the function in its most general form, as the use of definite integrals may, perhaps, involve an hypothesis which would be too restricted to represent all the bodies whose molecular forces act in the line joining the molecules.

The following relations will exist, in consequence of the form of the function,

$$\begin{aligned} \frac{dV}{du} &= \frac{dV}{d\beta_3} = \frac{dV}{d\gamma_2}; \\ \frac{dV}{dv} &= \frac{dV}{d\gamma_1} = \frac{dV}{d\alpha_3}; \\ \frac{dV}{dw} &= \frac{dV}{d\alpha_2} = \frac{dV}{d\beta_1}. \end{aligned} \tag{29}$$

These equations will reduce the equations of motion (5) to the following:

* Transactions of the Royal Irish Academy, vol. xxi. p. 151.

$$
\begin{aligned}
-\epsilon \frac{d^2\xi}{dt^2} &= \frac{d}{dx} \cdot \frac{dV}{d\alpha} + \frac{d}{dy} \cdot \frac{dV}{dw} + \frac{d}{dz} \cdot \frac{dV}{dv}; \\
-\epsilon \frac{d^2\eta}{dt^2} &= \frac{d}{dy} \cdot \frac{dV}{d\beta} + \frac{d}{dz} \cdot \frac{dV}{du} + \frac{d}{dx} \cdot \frac{dV}{dw}; \\
-\epsilon \frac{d^2\zeta}{dt^2} &= \frac{d}{dz} \cdot \frac{dV}{d\gamma} + \frac{d}{dx} \cdot \frac{dV}{dv} + \frac{d}{dy} \cdot \frac{dV}{du}.
\end{aligned} \qquad (30)
$$

These are equivalent to equations (24) of my former paper, and are an immediate consequence of (29), which express the relations among the resultants of the molecular forces consequent upon the restricted form of the function. Equations (29), or the corresponding equations (9),

$$Q_3 - R_2 = 0, \quad R_1 - P_3 = 0, \quad P_2 - Q_1 = 0,$$

are analogous to a statical theorem given by FRESNEL in his Memoir on double Refraction. If no external forces act upon the system, or if their influence may be assumed to be constant at all points of the body, then the coefficients of $V$ will become constants, and the linear part of $V$ will introduce no terms into the differential equations, which will in this case depend upon a homogeneous function of the second order. There will be, therefore, in general, twenty-one constants, viz., the coefficients of the function

$$
\begin{aligned}
2V = {} & (\alpha^2)\alpha^2 + (\beta^2)\beta^2 + (\gamma^2)\gamma^2 + (u^2)u^2 + (v^2)v^2 + (w^2)w^2 \\
& + 2\{(\beta\gamma)\beta\gamma + (\alpha\gamma)\alpha\gamma + (\alpha\beta)\alpha\beta\} + 2\{(vw)vw + (uw)uw + (uv)uv\} \\
& + 2u\{(\alpha u)\alpha + (\beta u)\beta + (\gamma u)\gamma\} + 2v\{(\alpha v)\alpha + (\beta v)\beta + (\gamma v)\gamma\} \\
& + 2w\{(\alpha w)\alpha + (\beta w)\beta + (\gamma w)\gamma\}.
\end{aligned}
$$

The equations of motion deduced from this form of the function $V$ are identical with the equations used by M. CAUCHY in his theory of light.* They are deduced by M. CAUCHY directly from the consideration of attractive and repulsive forces between the molecules. These equations have been used also by Mr. GREEN, who seems to have considered them as more general than M. CAUCHY's equations.†

The function (22) is not altered in form by an alteration of the co-ordinate

* Exercices de Mathematiques, tom. v., p. 19.

† Cambridge Philosophical Society's Transactions, vol. vii. p. 121.

axes, and is the most general which can be used for the case considered in the present section. The direction of the molecular vibration will not be either normal or transversal, but will be determined by the axes of an ellipsoid first noticed by M. CAUCHY in treating of bodies whose molecules attract and repel each other. It may be worth while to examine whether there are any subordinate functions which possess the property of not being changed in form by the change of co-ordinates; as such functions, if they exist, will contain the laws of classes of bodies contained under the general function (22). I have already discussed one such function (20), which is evidently a particular case of (22). I shall now determine another remarkable subordinate function, which, while it is much less general than (22), yet contains most of the bodies whose properties are at all interesting. The following formulæ of transformation may be easily demonstrated.*

$$\begin{aligned}
\alpha &= a^2\alpha' + b^2\beta' + c^2\gamma' + bcu' + acv' + abw';\\
\beta &= a'^2\alpha' + b'^2\beta' + c'^2\gamma' + b'c'u' + a'c'v' + a'b'w';\\
\gamma &= a''^2\alpha' + b''^2\beta' + c''^2\gamma' + b''c''u' + a''c''v' + a''b''w';\\
u &= 2a'a''\alpha' + 2b'b''\beta' + 2c'c''\gamma' + (b'c''+b''c')u' + (a'c''+a''c')v' + (a'b''+a''b')w';\\
v &= 2aa''\alpha' + 2bb''\beta' + 2cc''\gamma' + (bc''+b''c)u' + (ac''+a''c)v' + (ab''+a''b)w';\\
w &= 2aa'\alpha' + 2bb'\beta' + 2cc'\gamma' + (bc'+b'c)u' + (ac'+a'c)v' + (ab'+a'b)w'.
\end{aligned}$$

If we now assume six functions of $(\alpha, \beta, \gamma, u, v, w)$, defined by the following equations:

$$\begin{aligned}
\lambda &= u^2 - 4\beta\gamma, \quad \mu = v^2 - 4\alpha\gamma, \quad \nu = w^2 - 4\alpha\beta;\\
\phi &= 2\alpha u - vw, \quad \chi = 2\beta v - uw, \quad \psi = 2\gamma w - uv;
\end{aligned}$$

it may be shown without much difficulty, that these new functions are transformed by the following equations:

$$\begin{aligned}
\lambda &= a^2\lambda' + b^2\mu' + c^2\nu' + 2bc\phi' + 2ac\chi' + 2ab\psi';\\
\mu &= a'^2\lambda' + b'^2\mu' + c'^2\nu' + 2b'c'\phi' + 2a'c'\chi' + 2a'b'\psi';\\
\nu &= a''^2\lambda' + b''^2\mu' + c''^2\nu' + 2b''c''\phi' + 2a''c''\chi' + 2a''b''\psi'; \qquad (31)\\
\phi &= a'a''\lambda' + b'b''\mu' + c'c''\nu' + (b'c''+b''c')\phi' + (a'c''+a''c')\chi' + (a'b''+a''b')\psi';\\
\chi &= aa''\lambda' + bb''\mu' + cc''\nu' + (bc''+b''c)\phi' + (ac''+a''c)\chi' + (ab''+a''b)\psi';\\
\psi &= aa'\lambda' + bb'\mu' + cc'\nu' + (bc'+b'c)\phi' + (ac'+a'c)\chi' + (ab'+a'b)\psi'.
\end{aligned}$$

* Transactions of the Royal Irish Academy, vol. xxi. p. 160.

Hence a function of $(\lambda, \mu, \nu, \phi, \chi, \psi)$ will reproduce itself by transformation of co-ordinates; let the function be

$$2V = P\lambda + Q\mu + R\nu + 2F\phi + 2G\chi + 2H\psi.$$

I shall first prove the existence of three rectangular axes, for which this function reduces to its first three terms. If the axes of co-ordinates be transformed, and the coefficients of $(\phi', \chi', \psi')$ equated to zero, we obtain

$$Pbc + Qb'c' + Rb''c'' + F(b'c'' + b''c') + G(bc'' + b''c) + H(bc' + b'c) = 0;$$
$$Pac + Qa'c' + Ra''c'' + F(a'c'' + a''c') + G(ac'' + a''c) + H(ac' + a'c) = 0;$$
$$Pab + Qa'b' + Ra''b'' + F(a'b'' + a''b') + G(ab'' + a''b) + H(ab' + a'b) = 0.$$

These equations will be satisfied by assuming for axes of co-ordinates the axes of the ellipsoid

$$Px^2 + Qy^2 + Rz^2 + 2Fyz + 2Gxz + 2Hxy = 1.$$

Hence, for these particular axes,

$$2V = P(u^2 - 4\beta\gamma) + Q(v^2 - 4\alpha\gamma) + R(w^2 - 4\alpha\beta). \tag{32}$$

Using this value of $V$ in equations (30), we obtain for the equations of motion,

$$\begin{aligned} -\epsilon\frac{d^2\xi}{dt^2} &= R\frac{dZ}{dy} - Q\frac{dY}{dz}; \\ -\epsilon\frac{d^2\eta}{dt^2} &= P\frac{dX}{dz} - R\frac{dZ}{dx}; \\ -\epsilon\frac{d^2\zeta}{dt^2} &= Q\frac{dY}{dx} - P\frac{dX}{dy}: \end{aligned} \tag{33}$$

$(X, Y, Z)$ denoting the same functions as in (27). If the function were

$$2V = PX^2 + QY^2 + RZ^2, \tag{34}$$

equations (27) would be the same as (33).

These equations are those used by Professor MAC CULLAGH and by Mr. GREEN. They denote transverse vibrations, and will give FRESNEL's wave-surface for plane waves, and also the vibrations parallel to the plane of polarization. Mr. GREEN has deduced his equations from a homogeneous function of the second order of $(\alpha, \beta, \gamma, u, v, w)$, by restricting it so as to be capable of propagating only

normal and transverse vibrations; this restriction will reduce the constants from twenty-one to seven; six of which belong to the transverse vibrations, and the seventh to the normal vibration. It is to be remarked, however, that a function of $(\lambda, \mu, \nu, \phi, \chi, \psi)$ will represent only a part of the properties of bodies whose molecules attract and repel each other. Such a body is always capable of transmitting normal vibrations, and though it may transmit transverse vibrations following the laws of FRESNEL's wave-surface, yet the normal vibration cannot be supposed to vanish. If we include normal vibrations, the most general form of the subordinate function will be

$$V = F(\omega, \lambda, \mu, \nu, \phi, \chi, \psi). \tag{35}$$

The manner in which I have obtained equation (35) is not so direct as Mr. GREEN's method, and I have only used it for the sake of the intermediate equations (31), which exhibit a remarkable property of the quantities $(\lambda, \mu, \nu, \phi, \chi, \psi)$. The direct method of deducing it is the following. Let M. CAUCHY's ellipsoid (17) be constructed for the function $V$, which is homogeneous and of the second order with respect to $(\alpha, \beta, \gamma, u, v, w)$.

Equations (16) determine the directions of molecular vibration; in these equations, if $(l, m, n)$ be substituted for $(\cos \alpha, \cos \beta, \cos \gamma)$, we shall obtain the following:

$$\begin{aligned}
(Q' - R')\, mn + F'\,(n^2 - m^2) + H'ln - G'lm &= 0,\\
(R' - P')\, nl + G'\,(l^2 - n^2) + F'ml - H'mn &= 0,\\
(P' - Q')\, lm + H'\,(m^2 - l^2) + G'nm - F'nl &= 0.
\end{aligned}$$

These equations express that one of the axes of the ellipsoid is normal to the wave-plane, and consequently that the other two axes are contained in the wave-plane: by stating analytically that these conditions are true, independent of the position of the wave-plane, we obtain the following relations among the coefficients of $V$,

$$\begin{gathered}
(\beta u) = 0, \quad (\alpha v) = 0, \quad (\alpha w) = 0,\\
(\gamma u) = 0, \quad (\gamma v) = 0, \quad (\beta w) = 0,\\
(\alpha u) + 2(vw) = 0, \quad (\beta v) + 2(uw) = 0, \quad (\gamma w) + 2(uv) = 0,\\
(\alpha^2) = (\beta^2) = (\gamma^2) = 2(u^2) + (\beta\gamma) = 2(v^2) + (\alpha\gamma) = 2(w^2) + (\alpha\beta).
\end{gathered}$$

These fourteen equations will reduce the function $V$ to the form

$$2V = A\omega^2 + P\lambda + Q\mu + R\nu + 2F\phi + 2G\chi + 2H\psi. \qquad (35, \text{a})$$

This is the function used by Mr. GREEN, and is a particular case of equation (35). The first term of this function will determine the normal vibration, and the last six will represent, as we have seen, transverse vibrations propagated according to the laws of FRESNEL'S wave-surface.

If the body be homogeneous and uncrystalline, we shall have the relations,

$$P = Q = R,$$
$$F = 0, \quad G = 0, \quad H = 0,$$

which will reduce the function $V$ to the form

$$2V = A\omega^2 + P(\lambda + \mu + \nu). \qquad (36)$$

This function will represent homogeneous solids, liquids, and gases, and will be the complete function for these bodies, provided there be neither external forces nor pressures at the limits. If there be such forces, however, it will be necessary to add to the function (36), which is of the second order, other terms of the first order, as in equations (15). If the function (36) be assumed to represent all the forces engaged, the equations derived from it will represent the motion of a body abandoned to its own molecular actions, and freed from all external influence, such as gravitation, pressure of an atmosphere, &c. The known properties of solids, liquids, and gases, enable us to determine the form of the function (36), and thus lead to the terms to be added in the general case for each species of body.

It is generally admitted that a solid body, if abandoned to itself, would be capable of vibratory motion, and that its molecules, if displaced, would tend to return to their former position. A gas, if abandoned to itself, would be dissipated by the repulsive force of its molecules, so that in this case the function (36) should lead to an impossible result, as vibratory motion is impossible without the addition of pressures at the limits, or some equivalent forces. A liquid occupies a position intermediate between a solid and a gas; and if we assume that a liquid abandoned to itself will be in a state of unstable equilibrium (i. e. its molecules, if displaced, will not return to their original position, while, if undisturbed, they will not be dissipated), we shall obtain from (36)

the equations of liquid motion, including friction, which have been deduced by various writers from different considerations.

A liquid need not be supposed to be exactly in this state at all times; a slight cohesive or a slight repulsive force may be supposed to exist among its molecules, according to the quantity of caloric contained in it, or other physical circumstances, which may modify the intensity of the molecular actions. If such cohesive or repulsive forces be considered very small, as compared with the cohesive forces in a perfect solid, or the repulsive forces in a perfect gas, the equations deduced from the hypothesis, that these forces are zero, may still be used.

We obtain from equation (36) the following:

$$\frac{dV}{da} = A\omega - 2P(\omega - a), \qquad \frac{dV}{du} = Pu,$$

$$\frac{dV}{d\beta} = A\omega - 2P(\omega - \beta), \qquad \frac{dV}{dv} = Pv,$$

$$\frac{dV}{d\gamma} = A\omega - 2P(\omega - \gamma), \qquad \frac{dV}{dw} = Pw.$$

It is necessary and sufficient for stable equilibrium that these six forces should have signs contrary to the signs of ($\delta a$, $\delta\beta$, $\delta\gamma$, $\delta u$, $\delta v$, $\delta w$). If these be made positive, then the forces must have negative signs, and *vice versâ.*

Hence, for stable equilibrium, it is necessary that $A$ and $P$ be both negative, which will reduce the function (36) to the following,

$$-2V = A\omega^2 + P(\lambda + \mu + \nu). \tag{36, a}$$

Also the first three equations (changing the signs of $A$ and $P$), added together, must be negative; hence the condition,

$$(-3A + 4P)\,\omega < 0. \tag{36, b}$$

If the equilibrium be stable, $A$ cannot be less than $\frac{4P}{3}$; and if it be exactly equal to this value, we shall obtain the equations peculiar to liquids, because a displacement will produce no molecular force; and if $A < \frac{4P}{3}$, a molecular force

will be developed which will tend to increase the displacement. The function (36, a) will, therefore, represent solids, liquids, or gases, according as $A >, =, < \frac{4P}{3}$.

I shall consider, first, the equations of homogeneous solids. The function (36, a), substituted in equations (30), leads to the following equations of motion.

$$\begin{aligned}
\epsilon \frac{d^2\xi}{dt^2} &= (A - P)\frac{d\omega}{dx} + P\left(\frac{d^2\xi}{dx^2} + \frac{d^2\xi}{dy^2} + \frac{d^2\xi}{dz^2}\right), \\
\epsilon \frac{d^2\eta}{dt^2} &= (A - P)\frac{d\omega}{dy} + P\left(\frac{d^2\eta}{dx^2} + \frac{d^2\eta}{dy^2} + \frac{d^2\eta}{dz^2}\right), \qquad (37) \\
\epsilon \frac{d^2\zeta}{dt^2} &= (A - P)\frac{d\omega}{dz} + P\left(\frac{d^2\zeta}{dx^2} + \frac{d^2\zeta}{dy^2} + \frac{d^2\zeta}{dz^2}\right).
\end{aligned}$$

These equations of motion of solid bodies were first given by M. CAUCHY ;* equations identical in form, but with the relation $A = 3P$ between the coefficients, had been previously obtained by M. NAVIER.† M. POISSON deduced equations identical with those of M. NAVIER. Mr. GREEN has used the equations (37), with two independent constants, in his theory of light ;‡ and Mr. STOKES has recently called attention to the importance of retaining the two coefficients (leaving their ratio to be determined by experiment for each solid), in a memoir published in the Cambridge Philosophical Society's Transactions.§

The relation $A = 3P$ is a consequence of the use of definite integrals for the coefficients of the function $V$, and only represents a particular elastic solid; its introduction does not alter the form of equations (37), nor does it render them more simple than they are in their present state.

The additions necessary to be made to the equations of hydrodynamics, in order to take into account the friction of the fluid particles, have been given by many writers. M. NAVIER first stated the equations in their corrected form for incompressible fluids.‖ M. POISSON has treated of the subject in a me-

* Exercices des Mathematiques, tom. iii. p. 180.

† Memoires de l'Institut., tom. vii. p. 389.

‡ Transactions of Cambridge Philosophical Society, tom. vii. p. 11.

§ Vol. viii. part 3.

‖ Memoires de l'Institut., tom. vi. p. 414.

moir published in the *Journal de l'Ecole Polytechnique;** and, more recently, M. BARRÉ DE SAINT VENANT† and Mr. STOKES‡ have written on the friction of fluids in motion.

The quantities to be added to the ordinary equations of hydrodynamics are the right hand members of equations (37), introducing the relation $\left(A = \frac{4P}{3}\right)$, which expresses that a displacement produces no molecular force.

The equations of the motion of gases have been already given, on the supposition that there is no tangential action, which is equivalent to assuming that the vibrations are normal, and therefore $(P = 0)$. If we wish to take account of the friction of gases, we should use the equations which have just been indicated for liquid motion.

### SECTION V. COMPARISON OF MECHANICAL THEORIES OF LIGHT.

It is well known that different mechanical theories have been proposed to account for the phenomena of the movement of light in crystalline bodies, and that, although these theories differ in their fundamental hypotheses, yet, to some extent, they agree in representing the most obvious phenomena of double refraction. The laws of wave-propagation in crystals, are geometrical consequences of the properties of FRESNEL'S wave-surface; and no mechanical theory of light can be considered as even an approximation to the truth, unless it contains, as a deduction from its hypotheses, the wave-surface of FRESNEL. But it would be an error to conclude that any theory is correct, which satisfies this condition. There are, in fact, three different theories which satisfy this fundamental condition, and it is evident that they cannot all be true. The first of these theories was propounded by FRESNEL himself, in his memoir on Double Refraction.§ It is based on the hypothesis, that the luminiferous ether is composed of attracting and repelling molecules. The form of wave-surface known as FRESNEL'S is deduced by its author from this hypothesis, with the peculiarity that the vibrations of the molecules are perpendicular to the plane of polarization.

* Cahier xx. p. 139. † Comptes Rendus, tom. xvii. p. 1240.

‡ Transactions of Cambridge Philosophical Society, vol. viii. part 3.

§ Memoires de l'Institut, tom. vii., p. 45.

M. CAUCHY afterwards gave the general equations peculiar to such a system, and deduced from them FRESNEL'S wave-surface, as a first approximation to what he considered as the more accurate laws of wave-propagation.*

In the year 1839, Mr. GREEN presented to the Cambridge Philosophical Society a memoir, in which, by a modification of M. CAUCHY'S equations, he obtained FRESNEL'S wave-surface as an exact deduction from the theory.† This modification consists, as I have already stated, in restricting the system to propagate normal and transverse vibrations. In M. CAUCHY'S or Mr. GREEN'S theory, the vibration of the molecules is parallel to the plane of polarization. In the same year Professor MAC CULLAGH presented to this Academy a mechanical theory of light, not founded on the hypothesis of attracting and repelling molecules. The vibrations in this theory also are parallel to the plane of polarization, and the form of the wave-surface is that given by FRESNEL. These three theories of light, therefore, agree, so far as the laws of wave-propagation are concerned; and, excluding FRESNEL'S theory from the comparison (as the vibrations perpendicular to the plane of polarization make it distinct from the other two theories), there remain the mechanical theories of Mr. GREEN and Professor MAC CULLAGH, which are identical so far as the laws of wave propagation are concerned. The two theories are, however, really different in their fundamental assumptions; and this remarkable agreement in the laws of wave-propagation deduced from them admits of a simple explanation. I propose to account for the agreement, and to suggest the direction in which we should look for a true *experimentum crucis* between them.

The function $V$ used by Mr. GREEN, when reduced to its simplest form, will be

$$-2V = A\omega^2 + P\lambda + Q\mu + R\nu; \tag{38}$$

and the simplest form of Professor MAC CULLAGH'S equations will be derived from the function

$$-2V = PX^2 + QY^2 + RZ^2. \tag{39}$$

It is evident from what I have stated in the first section, that $(\lambda, \mu, \nu)$ will

* Memoires de l'Institut. tom. x., 1830.

† Transactions of Cambridge Philosophical Society, vol. vii. p. 121.

produce the same terms in the differential equations of motion, as $X^2$, $Y^2$, $Z^2$; for the squares will be the same in each, and the rectangles will be

$$(2\beta_3\gamma_2-4\beta_2\gamma_3,\ 2\gamma_1\alpha_3-4\alpha_1\gamma_3,\ 2\alpha_2\beta_1-4\alpha_1\beta_2),\ \text{and}\ (-2\beta_3\gamma_2,\ -2\gamma_1\alpha_3,\ -2\alpha_2\beta_1).$$

These two sets of rectangles will produce the same terms in the equations of motion, since we may transpose the differentiations without affecting the result, so far as the laws of propagation are concerned. The surfaces of wave-slowness deduced from (38) and (39) may be obtained immediately from equation (18). Equation (38) will give the following:

$$\begin{aligned} P' &= Al^2 + Rm^2 + Qn^2; & F' &= (A-P)\,mn;\\ Q' &= Am^2 + Pn^2 + Rl^2; & G' &= (A-Q)\,ln;\\ R' &= An^2 + Ql^2 + Pm^2; & H' &= (A-R)\,lm. \end{aligned} \tag{40}$$

And similarly from equations (39) will be found

$$\begin{aligned} P' &= Rm^2 + Qn^2; & F' &= -Pmn;\\ Q' &= Pn^2 + Rl^2; & G' &= -Qln;\\ R' &= Ql^2 + Pm^2; & H' &= -Rlm. \end{aligned} \tag{41}$$

These equations differ from the former only by not containing $A$.

The equation of wave-slowness (18) derived from (40) is

$$\begin{aligned} &\{A(x^2+y^2+z^2)-1\}\\ &\times\{(x^2+y^2+z^2)(QRx^2+PRy^2+PQz^2)-(Q+R)x^2-(P+R)y^2-(P+Q)z^2+1\}=0. \end{aligned} \tag{42}$$

The first factor of this equation represents a sphere whose radius is $\frac{1}{\sqrt{A}}$, and belongs to the normal vibration; the second factor is the equation of FRESNEL'S wave-surface, and in it the vibrations are transversal. The equation of wave-slowness deduced from Professor MAC CULLAGH'S function (39) will be the last factor of (42). So far, therefore, as the laws of wave-propagation are concerned, the functions (38) and (39) are equivalent, with this difference, that the function (38) introduces a normal wave, which does not enter into the equations derived from (39). It might be thought at first sight that we are at liberty to make $A = 0$, and thus reduce the function (38) to a function representing nothing but transverse vibrations; this, however, cannot be admitted, for as

(38) represents a body whose molecules act in the line joining them, a wave of normal compression is always possible. This will be rendered more evident by considering the conditions at the limits.

Let the limiting surface separating two bodies be the plane $(x, y)$, then the equations of condition (11) will become, for the function (38),

$$\begin{aligned} Q'v_0' &= Q''v_0''; & \xi_0' &= \xi_0''; \\ P'u_0' &= P''u_0''; & \eta_0' &= \eta_0''; \\ A'\omega_0' + Q'\alpha_0' + P'\beta_0' &= A''\omega_0'' + Q''\alpha_0'' + P''\beta_0''; & \zeta_0' &= \zeta_0''. \end{aligned} \tag{43}$$

These equations are equal in number to the unknown quantities, provided normal waves be included, because the unknown quantities of the problem are the intensities of the reflected and refracted waves; it is impossible, therefore, for exclusively transverse waves to be produced by reflexion or refraction in such a body as (38) defines: in order to obtain unknown quantities whose number shall be equal that of the necessary conditions of the mechanical problem, we must introduce normal vibrations. The conditions at the limits deduced from (39) are

$$\begin{aligned} Q'Y_0' &= Q''Y_0''; & \xi_0' &= \xi_0''; \\ P'X_0' &= P''X_0''; & \eta_0' &= \eta_0''; \\ & & \zeta_0' &= \zeta_0''. \end{aligned} \tag{44}$$

The additional hypothesis made by Professor MAC CULLAGH, that the density of the medium is the same in the two bodies, reduces these equations to four. Accordingly, with this hypothesis, there is no necessity to have recourse to normal waves, as there will be four intensities to be determined in the transverse waves.

From these considerations it appears, that the *experimenta crucis* between the rival theories of light must be sought for among the laws of reflexion and refraction; but unfortunately these laws are not known with sufficient accuracy to enable us to decide the question. Mr. GREEN's theory contains the common laws of reflexion at the surfaces of ordinary media as first approximations, while Professor MAC CULLAGH's system has the advantage of giving these laws as exact results; nothing, however, but more accurate experiments can decide whether the approximation or the exact result be most in accordance with the truth; and as these experiments involve considerations of the intensity of light, it would be

difficult to make them with sufficient accuracy. The present state of the wave theory of light certainly suggests grave doubts as to the nature of the foundation on which the whole system is based. We first assume the existence of an unknown medium, whose existence must remain unproved and unprovable by us; then, from supposed properties of this unknown medium, we deduce the laws of propagation, &c. Here a new difficulty arises; for we find several different theories capable of explaining the laws of propagation, and explaining with more or less exactness the most obvious of the laws of reflexion and refraction. How are we to decide among these conflicting theories? Are we to assume, with M. CAUCHY, that the observed laws of polarized light occupy, with respect to the mathematical laws deduced from his theory, the same position that the laws of KEPLER stand in with respect to the more accurate laws of planetary motion? or are we to assume that theory to be correct which agrees accurately with the common formulæ for reflected light, when it is well known that these formulæ themselves are doubtful for highly refracting substances? It appears certain, that we do not yet possess experimental knowledge sufficient to enable us to determine which of the theories of light is correct, or whether any of them be so. In a general point of view Professor MAC CULLAGH's theory possesses an important advantage, as compared with other theories. It contains no inexplicable normal wave, and does not render this difficult subject still more intricate, by the introduction of a useless vibration. It is greatly to be desired, that the attention of experimentalists were directed to the necessity which exists for more accurate and general researches into the laws of crystalline reflexion and refraction, and that the surface of FRESNEL were placed upon a purely experimental basis. From such researches, carefully conducted, might be deduced the geometrical laws of double refraction, and a foundation be laid for a complete and positive theory of the laws of polarized light.